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New Characterizations of Discrete Classical Orthogonal Polynomials

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We prove that if both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ are orthogonal polynomials for any fixed integer $r \geqslant 1$, then $\{P_n(x)\}_{n=0}^{\infty}$ must be discrete classical orthogonal polynomials. This result is a discrete version of the classical Hahn's theorem stating that if both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{(d/dx)^r P_n(x)\}_{n=r}^{\infty}$ are orthogonal polynomials, then $\{P_n(x)\}_{n=0}^{\infty}$ are classical orthogonal polynomials. We also obtain several other characterizations of discrete classical orthogonal polynomials.

1. INTRODUCTION

Consider a sequence of polynomials that arise as eigenfunctions of the second-order difference equation of hypergeometric type

$$L_2[y](x) = \ell_2(x)\, \Delta\nabla y(x) + \ell_1(x)\, \Delta y(x) = \lambda_n y(x), \tag{1.1}$$

where $\ell_2(x) = \ell_{22}x^2 + \ell_{21}x + \ell_{20}$ ($\not\equiv 0$) and $\ell_1(x) = \ell_{11}x + \ell_{10}$ are polynomials independent of n and

$$\lambda_n = n(n-1)\,\ell_{22} + n\ell_{11}, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

Orthogonal polynomials satisfying (1.1) are known as discrete classical orthogonal polynomials and they are well studied [6, 13, 15, 16, 19, 23]. Like classical orthogonal polynomials satisfying second-order differential equations of hypergeometric type, discrete classical orthogonal polynomials can be characterized in many different ways (see [1–5, 7, 8, 10, 14, 18]). In particular, it is well known that classical orthogonal polynomials (respectively, discrete classical orthogonal polynomials) are the only orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ such that $\{P'_n(x)\}_{n=1}^{\infty}$ (respectively, $\{\nabla P_n(x)\}_{n=1}^{\infty}$) is also orthogonal (see [4, 11, 12, 14, 17, 21, 22]). Later, Hahn [8] (see also [7, 9]) showed that the only orthogonal polynomials

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whose derivatives of any fixed order are also orthogonal are the classical orthogonal polynomials.

In this work, we obtain a discrete version of Hahn's theorem by showing that discrete classical orthogonal polynomials are the only orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ such that $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ (or $\{\Delta^r P_n(x)\}_{n=r}^{\infty}$) is quasi-orthogonal (see Definition 2.1) for any fixed integer $r \geqslant 1$.

2. PRELIMINARIES

All polynomials in this work are assumed to be real polynomials of a real variable x and we let $\mathscr{P}$ be the space of all polynomials. We denote the degree of a polynomial $\psi(x)$ by $\deg(\psi)$ with the convention that $\deg(0) = -1$.

By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geqslant 0$. We call any linear functional σ on $\mathscr{P}$ a moment functional and denote its action on a polynomial $\psi(x)$ by $\langle \sigma, \psi \rangle$. In particular, we call $\{\langle \sigma, x^n \rangle\}_{n=0}^{\infty}$ the moments of σ.

Any PS $\{P_n(x)\}_{n=0}^{\infty}$ determines a unique sequence of moment functionals $\{u_n\}_{n=0}^{\infty}$, called the dual sequence of $\{P_n(x)\}_{n=0}^{\infty}$ (cf. [18]), by the conditions

$$\langle u_n, P_m \rangle = \delta_{mn} \qquad (m \text{ and } n \geqslant 0), \tag{2.1}$$

where δ_{mn} is the Kronecker delta function. In particular, we call u_0 the canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

Definition 2.1. We call a PS $\{P_n(x)\}_{n=0}^{\infty}$ a quasi-orthogonal polynomial system (QOPS) (respectively, an orthogonal polynomial system (OPS)) if there is a non-zero moment functional σ such that

$$\langle \sigma, P_m P_n \rangle = K_n \delta_{mn} \qquad (m \text{ and } n \geqslant 0), \tag{2.2}$$

where K_n are real (respectively, non-zero real) constants. In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a QOPS or an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

Note that if $\{P_n(x)\}_{n=0}^{\infty}$ is a QOPS relative to σ, then $\langle \sigma, P_0^2 \rangle \neq 0$ but $\langle \sigma, P_n^2 \rangle$ for $n \geqslant 1$ may or may not be 0 and σ must be a non-zero constant multiple of the canonical moment functional u_0 of the PS $\{P_n(x)\}_{n=0}^{\infty}$.

We say that a moment functional σ is regular (respectively, positive-definite) if its moments $\{\langle \sigma, x^n \rangle\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\langle \sigma, x^{i+j} \rangle]_{i,j=0}^{n} \neq 0 \qquad (\text{respectively, } \Delta_n(\sigma) > 0) \tag{2.3}$$

for every $n \geqslant 0$. It is well known (see Chapter 1 in Chihara [5]) that a moment functional σ is regular if and only if there is an OPS relative to σ.

For a moment functional σ and a polynomial $\phi(x)$, we let $\Delta\sigma$, $\nabla\sigma$ and $\phi\sigma$, be the moment functionals defined by

$$\langle \Delta\sigma, \psi \rangle = -\langle \sigma, \nabla\psi \rangle, \qquad \langle \nabla\sigma, \psi \rangle = -\langle \sigma, \Delta\psi \rangle,$$
$$\langle \phi\sigma, \psi \rangle = \langle \sigma, \phi\psi \rangle \qquad (\psi \in \mathscr{P}),$$

where $\Delta\psi(x) = \psi(x+1) - \psi(x)$ and $\nabla\psi(x) = \psi(x) - \psi(x-1)$. Then we have the following Leibniz rule:

$$\Delta(\phi\sigma) = \phi(x+1)\, \Delta\sigma + \Delta(\phi)\, \sigma, \qquad \nabla(\phi\sigma) = \phi(x-1)\, \nabla\sigma + \nabla(\phi)\, \sigma, \tag{2.4}$$

and $\Delta\sigma = 0$ (or $\nabla\sigma = 0$) if and only if $\sigma = 0$.

LEMMA 2.1 [14]. *Let σ be a regular moment functional and $\{P_n(x)\}_{n=0}^{\infty}$ an OPS relative to σ. Then we have*

(i) *for any polynomial $\phi(x)$, $\phi(x)\,\sigma = 0$ if and only if $\phi(x) \equiv 0$.*

(ii) *for any moment functional τ $(\neq 0)$ and any integer $k \geqslant 0$, $\langle \tau, P_n \rangle = 0$ for $n > k$ if and only if $\tau = \psi(x)\,\sigma$ for some polynomial $\psi(x)$ of degree $\leqslant k$.*

In this case, $\deg(\psi) = k_0$ $(0 \leqslant k_0 \leqslant k)$ *is the largest integer such that $\langle \tau, P_n \rangle = 0$ for $n > k_0$ and $\langle \tau, P_{k_0} \rangle \neq 0$.*

LEMMA 2.2 [18]. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a PS and $\{u_n\}_{n=0}^{\infty}$ the dual sequence of $\{P_n(x)\}_{n=0}^{\infty}$. Then for any moment functional τ and any integer $k \geqslant 0$, the following two statements are equivalent.*

(i) *$\langle \tau, P_k \rangle \neq 0$ and $\langle \tau, P_n \rangle = 0$ for $n > k$.*

(ii) *There exist real constants $\{e_j\}_{j=0}^{k}$ such that $e_k \neq 0$ and*

$$\tau = \sum_{j=0}^{k} e_j u_j. \tag{2.5}$$

LEMMA 2.3. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a PS and $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ the dual sequences of PS's $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x) := (1/(n+1))\, \nabla P_{n+1}(x)\}_{n=0}^{\infty}$, respectively. Then, we have*

$$\Delta v_n = -(n+1)\, u_{n+1} \qquad (n \geqslant 0). \tag{2.6}$$

Proof. Since $\langle \Delta v_n, P_m \rangle = -\langle v_n, \nabla P_m \rangle = -m\langle v_n, Q_{m-1} \rangle = -m\delta_{n, m-1}$ for n and $m \geqslant 0$ $(Q_{-1}(x) \equiv 0)$, we have (2.6) by Lemma 2.2. ∎

LEMMA 2.4 [18]. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be a PS and* $\{u_n\}_{n=0}^{\infty}$ *the dual sequence of* $\{P_n(x)\}_{n=0}^{\infty}$. *Then the following two statements are equivalent.*

(i) $\{P_n(x)\}_{n=0}^{\infty}$ *is an OPS.*

(ii) *For each* $n \geqslant 0$, *there is a non-zero real constant* C_n *such that*

$$u_n = C_n P_n(x)\, u_0. \tag{2.7}$$

Note that Lemma 2.1 is an easy consequence of Lemma 2.3 and Lemma 2.4.

DEFINITION 2.2. An OPS $\{P_n(x)\}_{n=0}^{\infty}$ is called a discrete classical OPS if for each $n \geqslant 0$, $P_n(x)$ satisfies the second order difference equation (1.1).

PROPOSITION 2.5. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be an OPS relative to a regular moment functional* σ. *Then, the following statements are all equivalent.*

(i) $\{P_n(x)\}_{n=0}^{\infty}$ *is discrete classical OPS relative to* σ.

(ii) $\{\nabla P_n(x)\}_{n=1}^{\infty}$ *is an OPS.*

(iii) $\{\nabla P_n(x)\}_{n=1}^{\infty}$ *is a QOPS.*

(iv) *There are polynomials* $\ell_2(x)$ $(\not\equiv 0)$ *of degree* $\leqslant 2$ *and* $\ell_1(x)$ *of degree* 1 *such that* σ *satisfies*

$$\Delta(\ell_2 \sigma) = \ell_1 \sigma. \tag{2.8}$$

Proof. It is well known ([6, 19]) that (i) is equivalent to (iv).

(i) $\Rightarrow$ (ii): Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ satisfying the difference equation (1.1). At first we prove that $\lambda_n \neq 0$ for all $n \geqslant 1$. Assume $\lambda_n = 0$ for some $n \geqslant 1$. Then we have by (2.8)

$$\begin{aligned} 0 = \lambda_n P_n \sigma &= [\ell_2 \Delta\nabla P_n + \ell_1 \Delta P_n]\, \sigma \\ &= \ell_2 [\Delta\nabla P_n]\, \sigma + \Delta P_n \Delta(\ell_2 \sigma) \\ &= \Delta[(\nabla P_n)\, \ell_2 \sigma] \end{aligned}$$

so that $(\nabla P_n(x))\, \ell_2(x)\, \sigma = 0$. Hence $(\nabla P_n(x))\, \ell_2 \equiv 0$ by Lemma 2.1(i) and so $\nabla P_n(x) \equiv 0$ since $\ell_2(x) \not\equiv 0$, which implies $n = 0$ contradicting the fact that $n \geqslant 1$. Since (i) is equivalent to (iv), we have

$$\lambda_n P_n \sigma = \ell_2 \Delta\nabla P_n \sigma + \ell_1 \Delta P_n \sigma = \Delta[(\nabla P_n)\, \ell_2 \sigma].$$

Hence,

$$\begin{aligned} \langle \ell_2 \sigma, \nabla P_{m+1} \nabla P_{n+1} \rangle &= -\langle \Delta[(\nabla P_{n+1})\, \ell_2 \sigma], P_{m+1} \rangle \\ &= -\lambda_{n+1} \langle \sigma, P_{m+1} P_{n+1} \rangle. \end{aligned}$$

Therefore, $\{\nabla P_{n+1}(x)\}_{n=0}^{\infty}$ is an OPS relative to $\ell_2(x)\,\sigma$ since $\lambda_{n+1}\neq 0$, $n\geqslant 0$ and $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ.

Since (ii) implies (iii) by definition, it suffices to show that (iii) implies (iv).

(iii) $\Rightarrow$ (iv): Assume that $\{\nabla P_{n+1}(x)\}_{n=0}^{\infty}$ is a QOPS relative to τ $(\neq 0)$ so that

$$\langle \tau, \nabla P_{m+1} \nabla P_{n+1}\rangle = 0 \qquad \text{for} \quad m\neq n, \quad m \text{ and } n\geqslant 0. \tag{2.9}$$

Set $m=0$ in (2.9). Then we have for every $n>0$

$$0=\langle \tau, \nabla P_1 \nabla P_{n+1}\rangle = -\nabla P_1 \langle \Delta\tau, P_{n+1}\rangle$$

so that $\langle \Delta\tau, P_{n+1}(x)\rangle = 0$. Hence Lemma 2.1(ii) implies

$$\Delta\tau = \ell_1(x)\,\sigma \tag{2.10}$$

for some polynomial $\ell_1(x)$ of degree $\leqslant 1$. Set $m=1$ in (2.9). Then for every $n>1$, we have

$$\begin{aligned} 0 &= \langle \tau, \nabla P_2 \nabla P_{n+1}\rangle = -\langle \Delta[(\nabla P_2)\,\tau], P_{n+1}\rangle \\ &= -\langle [\Delta \nabla P_2]\,\tau, P_{n+1}\rangle - \langle \Delta P_2 \Delta\tau, P_{n+1}\rangle \\ &= -[\Delta\nabla P_2]\langle \tau, P_{n+1}\rangle - \langle \Delta\tau, [\Delta P_2]\,P_{n+1}\rangle \\ &= -[\Delta\nabla P_2]\langle \tau, P_{n+1}\rangle - \langle \sigma, \ell_1[\Delta P_2]\,P_{n+1}\rangle. \end{aligned}$$

Since $\langle \sigma, \ell_1[\Delta P_2]\,P_{n+1}\rangle = 0$ for $n>1$ and $\nabla\Delta P_2(x)\not\equiv 0$, $\langle \tau, P_{n+1}(x)\rangle = 0$ for $n>1$ so that by Lemma 2.1(ii),

$$\tau = \ell_2(x)\,\sigma \tag{2.11}$$

for some polynomial $\ell_2(x)$ of degree $\leqslant 2$. The equation (2.8) follows from (2.10) and (2.11) and $\ell_1(x)\not\equiv 0$, $\ell_2(x)\not\equiv 0$ since $\tau\neq 0$. If $\ell_1(x)=c$, c a non-zero constant, then

$$\langle \sigma, 1\rangle = \frac{1}{c}\langle \Delta(\ell_2\sigma), 1\rangle = 0,$$

which is impossible since σ is regular. Hence, $\deg(\ell_1)=1$. ∎

Remark 2.1. In fact, Hahn ([9]) proved the equivalence of the statements (i) and (ii) in Proposition 2.5 in more general setting. He first introduced a linear operator

$$Lf(x) = \frac{f(qx+w)-f(x)}{(q-1)\,x+w},$$

where q and w are given constants, and then characterized all OPS's $\{P_n(x)\}_{n=0}^{\infty}$ such that $\{LP_n(x)\}_{n=1}^{\infty}$ is also an OPS. Note that when $q=1$ and $w=1$ or -1, L becomes Δ or ∇ respectively and when $w=0$ and $q\to 1$, L becomes d/dx.

As an immediate consequence of Proposition 2.5, we obtain: if $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS satisfying the difference equation (1.1), then $\{\nabla P_n(x)\}_{n=1}^{\infty}$ is also a discrete classical OPS satisfying the difference equation

$$\ell_2(x-1)\,\Delta\nabla y(x)+(\nabla\ell_2(x)+\ell_1(x))\,\Delta y(x)=(\lambda_n-\nabla\ell_1(x))\,y(x).$$

By induction, for any integer $r\geqslant 1$, $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ is also a discrete classical OPS.

Definition 2.3 [20]. A moment functional σ is called discrete semi-classical if σ is regular and there are polynomials $\phi(x)\not\equiv 0$ and $\psi(x)$ of degree $\geqslant 1$ such that

$$\Delta(\phi\sigma)=\psi\sigma. \tag{2.12}$$

For any discrete semi-classical moment functional σ, we call $s:=\min\{\max(\deg(\phi)-2, \deg(\psi)-1\}$ the class number of σ, where the minimum is taken over all pairs of polynomials (ϕ, ψ) satisfying the equation (2.12). In this case, we call σ a discrete semi-classical moment functional of class s and an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ is called a discrete semi-classical OPS of class s.

We can restate the equivalence of the statements (i) and (iv) in Proposition 2.5 as: an OPS is a discrete classical OPS if and only if it is a discrete semi-classical OPS of class 0.

Lemma 2.6. *Let σ be a discrete semi-classical moment functional satisfying*

$$\begin{aligned}\Delta(\phi_1\sigma)&=\psi_1\sigma \qquad (s_1:=\max(t_1-2,\, p_1-1))\\ \Delta(\phi_2\sigma)&=\psi_2\sigma \qquad (s_2:=\max(t_2-2,\, p_2-1)),\end{aligned} \tag{2.13}$$

where $t_j=\deg(\phi_j)$ and $p_j=\deg(\psi_j)$, $j=1,2$. Let $\phi(x)$ be a common factor of $\phi_1(x)$ and $\phi_2(x)$ of the highest degree. Then, there is a polynomial $\psi(x)$ such that

$$\Delta(\phi\sigma)=\psi\sigma,$$

where $s:=\max(\deg(\phi)-2, \deg(\psi)-1)=s_1-t_1+\deg(\phi)=s_2-t_2+\deg(\phi)$.

Proof. We may assume that $\phi_1 = \tilde{\phi}_1 \phi$ and $\phi_2 = \tilde{\phi}_2 \phi$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are co-prime polynomials. From the equation (2.13), we have

$$\tilde{\phi}_1(x+1)\, \Delta(\phi\sigma) = (\psi_1 - \phi \Delta \tilde{\phi}_1)\, \sigma, \tag{2.14}$$

$$\tilde{\phi}_2(x+1)\, \Delta(\phi\sigma) = (\psi_2 - \phi \Delta \tilde{\phi}_2)\, \sigma. \tag{2.15}$$

Multiplying (2.14) by $\tilde{\phi}_2(x+1)$ and (2.15) by $\tilde{\phi}_1(x+1)$ and substracting the resulting two equations, we have

$$(\psi_1 - \phi \Delta \tilde{\phi}_1)\, \tilde{\phi}_2(x+1) = (\psi_2 - \phi \Delta \tilde{\phi}_2)\, \tilde{\phi}_1(x+1).$$

Since $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are co-prime, $\tilde{\phi}_1(x+1)$ and $\tilde{\phi}_2(x+1)$ are also co-prime. Hence $\psi_2 - \phi \Delta \tilde{\phi}_2$ and $\psi_1 - \phi \Delta \tilde{\phi}_1$ are divisible by $\tilde{\phi}_2(x+1)$ and $\tilde{\phi}_1(x+1)$ respectively so that there exists a polynomial ψ such that

$$\psi_2 - \phi \Delta \tilde{\phi}_2 = \psi \tilde{\phi}_2(x+1) \qquad \text{and} \qquad \psi_1 - \phi \Delta \tilde{\phi}_1 = \psi \tilde{\phi}_1(x+1).$$

From the equation (2.14) and (2.15), we have

$$\tilde{\phi}_2(x+1)[\Delta(\phi\sigma) - \psi\sigma] = 0 \qquad \text{and} \quad \tilde{\phi}_1(x+1)[\Delta(\phi\sigma) - \psi\sigma] = 0.$$

Since $\tilde{\phi}_1(x+1)$ and $\tilde{\phi}_2(x+1)$ are co-prime, we have another equation of the form (2.12):

$$\Delta(\phi\sigma) - \psi\sigma = 0.$$

The class number follows from just counting degrees of $\phi(x)$ and $\psi(x)$. ∎

PROPOSITION 2.7. *Let σ be a discrete semi-classical moment functional of class s satisfying the equation* (2.12) *with* $s = \max(\deg(\phi) - 2, \deg(\psi) - 1)$. *If σ satisfies the equation* (2.12) *with another pair of polynomials* $(\phi_1, \psi_1) \neq (0, 0)$, *then $\phi_1(x)$ is divisible by $\phi(x)$.*

Proof. Let $\alpha(x)$ be the greatest common divisor of $\phi(x)$ and $\phi_1(x)$. Then by Lemma 2.6, there is a polynomial $\beta(x)$ such that

$$\Delta(\alpha\sigma) = \beta\sigma$$

and $s_0 := \max(\deg(\alpha) - 2, \deg(\beta) - 1) = s - \deg(\phi) + \deg(\alpha)$. Since $s_0 \geqslant s$, $\deg(\alpha) \geqslant \deg(\phi)$ so that $\alpha(x) = c\phi(x)$ for some non-zero constant c. Hence, $\phi(x)$ must divide $\phi_1(x)$. ∎

Remark 2.2. The continuous versions of Lemma 2.6 and Proposition 2.7 are proved in [18] and [14] respectively.

3. MAIN THEOREMS.

We start with a theorem.

THEOREM 3.1. *For an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *relative to a regular moment functional* σ *and an integer* $r \geqslant 1$, *the following statements are all equivalent.*

(i) $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ *is a QOPS.*

(ii) *There are* $r+1$ *polynomials* $\{a_k(x)\}_r^{2r}$ *with* $a_{2r}(x) \not\equiv 0$, $\deg(a_k) \leqslant k$, $k = r, r+1, ..., 2r$, *and*

$$\Delta(a_k \sigma) = a_{k-1} \sigma, \qquad k = r+1, ..., 2r. \tag{3.1}$$

(iii) *There are moment functional* τ $(\neq 0)$ *and* $r+1$ *polynomials* $\{a_k(x)\}_r^{2r}$ *with* $\deg(a_k) \leqslant k$, $k = r, r+1, ..., 2r$ *and*

$$\Delta^{2r-k} \tau = a_k(x)\, \sigma, \qquad k = r, r+1, ..., 2r. \tag{3.2}$$

Proof. (i) $\Rightarrow$ (iii): Assume that $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ is a QOPS relative to τ $(\neq 0)$. Then,

$$\langle \tau, \nabla^r P_r \rangle \neq 0 \qquad \text{and} \qquad \langle \tau, \nabla^r P_m \nabla^r P_n \rangle = 0 \qquad \text{for all} \quad m \neq n.$$

For $m = r$, we have $\langle \tau, \nabla^r P_n \rangle = (-1)^r \langle \Delta^r \tau, P_n \rangle = 0$ for all $n \geqslant r+1$ so that by Lemma 2.1(ii),

$$\Delta^r \tau = a_r(x)\, \sigma \qquad \text{with} \quad \deg(a_r) \leqslant r.$$

In fact, $\deg(a_r) = r$ since $\langle \Delta^r \tau, P_r \rangle \neq 0$. For $m = r+1$, we have for any $n \geqslant r+2$,

$$\begin{aligned} 0 &= \langle \tau, \nabla^r P_{r+1} \nabla^r P_n \rangle = (-1)^r \langle \Delta^r [(\nabla^r P_{r+1})\, \tau], P_n \rangle \\ &= (-1)^r \langle \Delta^{r-1} [\nabla^r P_{r+1}(x+1)\, \Delta\tau + (\Delta \nabla^r P_{r+1})\, \tau], P_n \rangle \\ &= (-1)^r \langle \Delta^r P_{r+1} \Delta^r \tau + c(r)\, \Delta^{r-1} \tau, P_n \rangle, \end{aligned}$$

where the constant $c(r) = \Delta^{r+1}[P_{r+1}(x-r+1) + P_{r+1}(x-r+2) + \cdots + P_{r+1}(x)] = r(r+1)!$. Hence, we have by Lemma 2.1(ii)

$$\Delta^r P_{r+1} \Delta^r \tau + c(r)\, \Delta^{r-1} \tau = \phi_{r+1} \sigma,$$

where $\deg(\phi_{r+1}) \leqslant r+1$. Hence, $\Delta^{r-1} \tau = a_{r+1} \sigma$ with $\deg(a_{r+1}) = \deg(\phi_{r+1} - \Delta^r P_{r+1} a_r) \leqslant r+1$. Continuing the same process for $m = r+2, r+3, ..., 2r$, we obtain (iii).

(ii) $\Leftrightarrow$ (iii): It immediately follows by taking $\tau = a_{2r}(x) \sigma$.

(iii) $\Rightarrow$ (i): Assume that the condition (iii) holds. Then we have for $r \leqslant m < n$

$$
\begin{aligned}
\langle \tau, \nabla^r P_m \nabla^r P_n \rangle &= (-1)^r \langle \Delta^r [(\nabla^r P_m)\, \tau], P_n \rangle \\
&= (-1)^r \langle \Delta^{r-1} [(\Delta \nabla^r P_m)\, \tau + (\Delta \nabla^{r-1} P_m)\, \Delta \tau], P_n \rangle \\
&= (-1)^r \left\langle \sum_{k=0}^{r} \binom{r}{k} (\Delta^r \nabla^{r-k} P_m)\, \Delta^k \tau, P_n \right\rangle \\
&= (-1)^r \sum_{k=0}^{r} \binom{r}{k} \langle \Delta^k \tau, (\Delta^r \nabla^{r-k} P_m)\, P_n \rangle \\
&= (-1)^r \sum_{k=0}^{r} \binom{r}{k} \langle \sigma, a_{2r-k} (\Delta^r \nabla^{r-k} P_m)\, P_n \rangle \\
&= 0,
\end{aligned}
$$

since $\deg(a_{2r-k}[\Delta^r \nabla^{r-k} P_m]) \leqslant m$. Hence $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ is a QOPS relative to τ. ∎

Remark 3.1. For arbitrary constant a $(\neq 0)$ and b, we have that $\{P_n(ax+b)\}_{n=0}^{\infty}$ is also an OPS if $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS. Hence, the condition (i) in Theorem 3.1 is equivalent to that $\{\Delta^r P_n(x) = \nabla^r P_n(x+r)\}_{n=r}^{\infty}$ is a QOPS. In fact, we have the same results even though Δ or ∇ in Proposition 2.5 and in Theorem 3.1 are replaced by ∇ or Δ respectively.

LEMMA 3.2 (cf. Lemma 3.4 in [14]). *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a monic OPS relative to a regular moment functional σ. For an integer $r \geqslant 1$, let $\{Q_n(x) := (1/(P(n+r-1, r-1)))\, \nabla^{r-1} P_{n+r-1}(x)\}_{n=0}^{\infty}$ and $\{R_n(x) := (1/(n+1))\, \nabla Q_{n+1}(x)\}_{n=0}^{\infty}$. If $\{R_n(x)\}_{n=0}^{\infty}$ is a QOPS relative to τ, then $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the following recurrence relation:*

$$
Q_{n+1}(x) = (x - \beta_n)\, Q_n(x) - \gamma_n Q_{n-1}(x) - \sum_{j=0}^{n-2} \delta_n^j Q_j(x), \qquad n \geqslant 1, \tag{3.3}
$$

where β_n, γ_n, and δ_n^j are real constants with $\delta_1^0 = \delta_1^{-1} = 0$ and $\delta_n^1 = 0$, $n \geqslant 1$.

Proof. Since $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a three-term recurrence relation

$$
P_{n+1}(x) = (x - b_n)\, P_n(x) - c_n P_{n-1}(x), \qquad n \geqslant 1, \tag{3.4}
$$

where b_n and c_n are real constants with $c_n \neq 0$, $n \geqslant 1$. Replacing n by $n+r-1$ in (3.4) and then acting ∇^{r-1} and ∇^r on both sides, we obtain for $n \geqslant 0$

$$\nabla^{r-1} P_{n+r}(x) = (x - r + 1 - b_{n+r-1}) \nabla^{r-1} P_{n+r-1}(x) - c_{n+r-1} \nabla^{r-1} P_{n+r-2}(x) + (r-1) \nabla^{r-2} P_{n+r-1}(x), \quad (3.5)$$

$$\nabla^r P_{n+r}(x) = (x - r - b_{n+r-1}) \nabla^r P_{n+r-1}(x) - c_{n+r-1} \nabla^r P_{n+r-2}(x) + r \nabla^{r-1} P_{n+r-1}(x). \quad (3.6)$$

On the other hand, as a monic PS, $\{R_n(x)\}_{n=0}^{\infty}$ satisfy

$$R_{n+1}(x) = (x - \tilde{b}_n) R_n(x) - \tilde{c}_n R_{n-1}(x) - \sum_{j=0}^{n-2} \tilde{\delta}_n^j R_j(x), \qquad n \geqslant 1, \quad (3.7)$$

where $\tilde{b}_n$, $\tilde{c}_n$, and $\tilde{\delta}_n^j$ are real constants with $\tilde{\delta}_1^0 = \tilde{\delta}_1^{-1} = 0$ and $R_{-1}(x) \equiv 0$. Applying τ to (3.7) and using the quasi-orthogonality of $\{R_n(x)\}_{n=0}^{\infty}$ relative to τ, we obtain $\tilde{\delta}_n^0 = 0$, $n \geqslant 2$ so that (3.7) reduces to

$$R_{n+1}(x) = (x - \tilde{b}_n) R_n(x) - \tilde{c}_n R_{n-1}(x) - \sum_{j=1}^{n-2} \tilde{\delta}_n^j R_j(x), \qquad n \geqslant 2 \quad (\tilde{\delta}_2^1 = 0). \quad (3.8)$$

From (3.8) with n replaced by $n + r - 1$ and (3.6), we obtain

$$\begin{aligned} r \nabla^{r-1} P_{n+r-1} &= \left[\frac{r}{n} x + r + b_{n+r-1} - \tilde{b}_{n-1} \frac{n+r}{n} \right] \nabla^r P_{n+r-1} \\ &\quad + \left[c_{n+r-1} - \tilde{c}_{n-1} \frac{(n+r)(n+r-1)}{n(n-1)} \right] \nabla^r P_{n+r-1} \\ &\quad - \sum_{j=1}^{n-3} \tilde{\delta}_{n-1}^j \frac{(n+1)_r}{(j+1)_r} \nabla^r P_{j+r}, \\ &= \nabla \left[\left(\frac{r}{n} x + r + b_{n+r-1} - \tilde{b}_{n-1} \frac{n+r}{n} \right) \nabla^{r-1} P_{n+r-1} \right] \\ &\quad + \frac{r}{n} \nabla^r P_{n+r-1} - \frac{r}{n} \nabla^{r-1} P_{n+r-1} \\ &\quad + \left[c_{n+r-1} - \tilde{c}_{n-1} \frac{(n+r)(n+r-1)}{n(n-1)} \right] \nabla^r P_{n+r-2} \\ &\quad - \sum_{j=1}^{n-3} \tilde{\delta}_{n-1}^j \frac{(n+1)_r}{(j+1)_r} \nabla^r P_{j+r}, \qquad n \geqslant 3. \end{aligned}$$

Since, for any polynomials $f(x)$ and $g(x)$, $\nabla g(x) = \nabla f(x)$ if and only if $f(x) = g(x) + c$ with arbitrary constant c, we have

$$\nabla^{r-2} P_{n+r-1} = \left(\frac{x}{n+1} + \frac{n b_{n+r-1}}{r(n+1)} - \frac{(n+r)\,\tilde{b}_{n-1}}{r(n+1)} - \frac{n+r}{n+1} \right) \nabla^{r-1} P_{n+r-1}$$
$$+ \left(\frac{n c_{n+r-1}}{r(n+1)} - \frac{(n+r)(n+r-1)\,\tilde{c}_{n-1}}{r(n-1)(n+1)} \right) \nabla^{r-1} P_{n+r-2}$$
$$- \sum_{j=1}^{n-3} \tilde{\delta}_{n-1}^{j} \frac{n}{r(n+1)} \frac{(n+1)_r}{(j+1)_r} \nabla^{r-1} P_{j+r} + d_n, \tag{3.9}$$

where d_n is a constant. Substituting (3.9) into (3.5) yields

$$\nabla^{r-1} P_{n+r} = \left(\frac{-(n+r)\,c_{n+r-1}}{(n+1)\,r} - \frac{(r-1)(n+r)(n+r-1)\,\tilde{c}_{n-1}}{r(n-1)(n+1)} \right)$$
$$\times \nabla^{r-1} P_{n+r-2} + \frac{n+r}{r(n+1)}$$
$$\times \left(rx - b_{n+r-1} - (r-1)\,\tilde{b}_{n-1} + \frac{r(1-r)(n+r+1)}{n+r} \right)$$
$$\times \nabla^{r-1} P_{n+r-1} - \sum_{j=1}^{n-3} \frac{(r-1)\,n}{r(n+1)} \frac{(n+1)_r}{(j+1)_r}$$
$$\times \nabla^{r-1} P_{j+r} + (r-1)\,d_n, \qquad n \geqslant 3.$$

This last equation can be rewritten into the equation (3.3) by the definition of $Q_n(x)$ for $n \geqslant 3$. The equation (3.3) for $n=1$ or 2 is trivial. ∎

LEMMA 3.3 (cf. Lemma 3.5 in [14]). *Let* $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, *and* $\{R_n(x)\}_{n=0}^{\infty}$ *be the same as in Lemma* 3.2. *Let* $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, *and* $\{w_n\}_{n=0}^{\infty}$ *be the dual sequences of* $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, *and* $\{R_n(x)\}_{n=0}^{\infty}$ *respectively. If* $\{R_n(x)\}_{n=0}^{\infty}$ *is a QOPS, then*

(i) *there are* $r+1$ *polynomials* $\{a_k(x)\}_r^{2r}$ *with* $a_{2r}(x) \not\equiv 0$, $\deg(a_k) \leqslant k$, $k = r, ..., 2r$, *and*

$$\Delta^{2r-k} w_0 = a_k(x)\, u_0, \qquad k = r, ..., 2r \tag{3.10}$$

and

(ii) *there are* r *polynomials* $\{h_k(x)\}_{r+1}^{2r}$ *with* $h_{2r}(x) \not\equiv 0$, $\deg(h_k) \leqslant k$, $k = r+1, ..., 2r$, *and*

$$\Delta^{2r-k} v_0 = h_k(x)\, u_0, \qquad k = r+1, ..., 2r. \tag{3.11}$$

Moreover, we also have $\deg(a_r) = r$ *and* $\deg(h_{r+1}) = r-1$.

Proof. Assume that $\{R_n(x)\}_{n=0}^{\infty}$ is a QOPS. Then w_0 is an orthogonalizing moment functional of $\{R_n(x)\}_{n=0}^{\infty}$. Hence we have (i) from the equivalence of the statements (i) and (iii) in Theorem 3.1.

By Lemma 3.2, $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the recurrence relation (3.3). Applying v_1 to (3.3), we obtain $\langle xv_1, Q_n\rangle = 0$, $n \geqslant 3$ so that by Lemma 2.2

$$xv_1 = e_0 v_0 + e_1 v_1 + e_2 v_2,$$

where $e_j = \langle xv_1, Q_j\rangle$, $j = 0, 1, 2$. Since $e_0 = \langle xv_1, Q_0\rangle = \langle v_1, x\rangle = \langle v_1, Q_1\rangle = 1$, we have by Lemma 2.3

$$v_0 = (-x + e_1)\, \Delta w_0 + \frac{e_2}{2}\, \Delta w_1. \tag{3.12}$$

On the other hand, applying w_0 to (3.8), we obtain $\langle xw_0, R_n\rangle = 0$, $n \geqslant 2$ so that by Lemma 2.2

$$xw_0 = c_0 w_0 + c_1 w_1,$$

where $c_j = \langle xw_0, R_j\rangle$, $j = 0, 1$. If $c_1 = 0$, then $(x - c_0)\, w_0 = (x - c_0)\, a_{2r}(x)\, u_0 = 0$ by (3.10). It is a contradiction since u_0 is regular and $a_{2r}(x) \not\equiv 0$. Hence, $c_1 \neq 0$ and

$$w_1 = \frac{x - c_0}{c_1}\, w_0. \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$v_0 = \pi_{2r}(x)\, u_0, \tag{3.14}$$

where $\pi_{2r}(x)$ is a polynomial of degree $\leqslant 2r$. Acting Δ on (3.14) successively, we obtain (3.11) from (3.10).

Finally we have

$$\begin{aligned}\langle \Delta^r w_0, P_n\rangle &= (-1)^r \langle w_0, \nabla^r P_n\rangle \\ &= \begin{cases} 0 & \text{if } n \neq r \\ (-1)^r \langle w_0, P(n, r)\, R_{n-r}\rangle & \text{if } n = r \end{cases}\end{aligned}$$

so that $a_r(x)\, u_0 = \Delta^r w_0 = (-1)^r\, r!\, u_r = (-1)^r\, r!\, C_r P_r(x)\, u_0$ by Lemma 2.4. Hence $\deg(a_r) = r$.

Similarly we have $h_{r+1}(x)\, u_0 = \Delta^{r-1} v_0 = (-1)^{r-1}\, (r-1)!\; C_{r-1} P_{r-1}(x)\, u_0$ so that $\deg(h_{r+1}) = r - 1$. ∎

Now, we are ready to give our main result which is the discrete version of Hahn's theorem [8].

THEOREM 3.4. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to a regular moment functional σ and $r \geqslant 1$ an integer. Then any one of the equivalent statements in Theorem* 3.1 *is also equivalent to*

(iv) $\{P_n(x)\}_{n=0}^{\infty}$ *is a discrete classical OPS.*

Proof. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS. Then, $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ is also a discrete classical OPS for any integer $r \geqslant 1$. Hence, the statement (i) in Theorem 3.1 holds.

Conversely, we assume that the statement (i) in Theorem 3.1 holds. If $r=1$, $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS by the Proposition 2.5. Hence we assume $r \geqslant 2$. Then, by induction, it suffices to show that $\{\nabla^{r-1} P_n(x)\}_{r-1}^{\infty}$ is a QOPS or equivalently there exist $r-1$ polynomials $\{g_k(x)\}_{r-1}^{2(r-1)}$ with $g_{2(r-1)}(x) \not\equiv 0$, $\deg(g_k) \leqslant k$, $r-1 \leqslant k \leqslant 2(r-1)$ and

$$\Delta(g_k \sigma) = g_{k-1} \sigma, \qquad k = r, r+1, ..., 2r-2.$$

We may assume $\{P_n(x)\}_{n=0}^{\infty}$ is a monic PS and let $\{Q_n(x)\}_{n=0}^{\infty}$, $\{R_n(x)\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, and $\{w_n\}_{n=0}^{\infty}$ be the same as in Lemma 3.3. Since $\{\nabla^r P_n(x)\}_{n=r}^{\infty}$ is a QOPS, by Lemma 3.3, we have polynomials $\{a_k(x)\}_r^{2r}$ and $\{h_k(x)\}_{r+1}^{2r}$ satisfying (3.10) and (3.11). Hence, the moment functional u_0 satisfies

$$\Delta(a_k u_0) = a_{k-1} u_0, \qquad k = r+1, ..., 2r, \tag{3.15}$$

$$\Delta(h_k u_0) = h_{k-1} u_0, \qquad k = r+2, ..., 2r. \tag{3.16}$$

Now, let s ($\geqslant 0$) be the class number of the discrete semi-classical moment functional u_0 and $(\alpha(x), \beta(x)) \neq (0, 0)$ a pair of polynomials satisfying

$$\Delta(\alpha u_0) = \beta u_0 \qquad \text{with} \quad s = \max(\deg(\alpha) - 2, \deg(\beta) - 1).$$

Then we have from Proposition 2.7

$$a_k(x) = \tilde{a}_k(x)\, \alpha(x), \qquad k = r+1, ..., 2r, \tag{3.17}$$

$$h_k(x) = \tilde{h}_k(x)\, \alpha(x), \qquad k = r+2, ..., 2r, \tag{3.18}$$

where $\tilde{a}_k(x)$ and $\tilde{h}_k(x)$ are polynomials. Hence we have from (3.15), (3.16), (3.17), and (3.18)

$$\Delta \tilde{a}_k \alpha + \tilde{a}_k(x+1)\, \beta = a_{k-1}, \qquad k = r+1, ..., 2r; \tag{3.19}$$

$$\Delta \tilde{h}_k \alpha + \tilde{h}_k(x+1)\, \beta = h_{k-1}, \qquad k = r+2, ..., 2r. \tag{3.20}$$

From now on, we divide the proof into two cases: $s = \deg(\alpha) - 2 \geqslant \deg(\beta) - 1$ and $s = \deg(\beta) - 1 > \deg(\alpha) - 2$.

Case I. $s=\deg(\alpha)-2\geqslant\deg(\beta)-1$. Counting degrees on both sides of the equation (3.19), we have $\deg(a_{k-1})+1\leqslant\deg(a_k)$, $r+1\leqslant k\leqslant 2r$ since $\deg(\alpha)\geqslant\deg(\beta)+1$. Hence we have

$$\deg(a_k)=k, \qquad k=r, r+1, ..., 2r$$

since $\deg(a_r)=r$ and $\deg(a_k)\leqslant k$, $k=r, ..., 2r$. Similarly, counting degrees on both sides of the equation (3.20), we have $\deg(h_{k-1})+1\leqslant\deg(h_k)$, $r+2\leqslant k\leqslant 2r$. We now claim that

$$\deg(h_{k-1})+1=\deg(h_k), \qquad k=r+2, ..., 2r. \tag{3.21}$$

If not, let j be the first integer $\geqslant r+2$ such that $\deg(h_{j-1})+1<\deg(h_j)$. Then, $\deg(h_k)=k-2$, $k=r+1, ..., j-1$ and $j-2<\deg(h_j)\leqslant j$ since $\deg(h_{r+1})=r-1$. Since $r+2\leqslant j\leqslant 2r$, $\deg(h_j)=m=\deg(a_m)$ for some $m=r+1, ..., 2r$. Let A ($\neq 0$) and B ($\neq 0$) be the leading coefficients of $a_m(x)$ and $h_j(x)$ respectively. Multiplying the equation (3.19) for $k=m$ by B and the equation (3.20) for $k=j$ by A and subtracting these two equations, we obtain

$$\begin{aligned}(B\Delta\tilde{a}_m-A\Delta\tilde{h}_j)\,\alpha+(B\tilde{a}_m(x+1)-A\tilde{h}_j(x+1))\,\beta\\ =Ba_{m-1}-Ah_{j-1}.\end{aligned} \tag{3.22}$$

We then have $\deg(Ba_{m-1}-Ah_{j-1})=m-1$ since $\deg(a_{m-1})=m-1>j-3=\deg(h_{j-1})$. However, the degree of the left hand side of the equation (3.22) is at most $m-2$ since $\deg(Ba_m-Ah_j)\leqslant m-1$ and $\deg(\beta)\leqslant\deg(\alpha)-1$. It is a contradiction so that we have (3.21).

Since $\deg(h_{r+1})=r-1$, we have from (3.21)

$$\deg(h_k)=k-2, \qquad k=r+1, ..., 2r. \tag{3.23}$$

If we set $g_k(x)=h_{k+2}(x)$, $k=r-1, ..., 2(r-1)$, then $\{g_k\}_{r-1}^{2(r-1)}$ satisfy the condition (ii) in Theorem 3.1 with r replaced by $r-1$ and σ replaced by u_0 by (3.16) and (3.23). Hence, by Theorem 3.1 and induction hypothesis, $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS relative to u_0.

Case II. $s=\deg(\beta)-1>\deg(\alpha)-2$. Counting degrees on both sides of the equation (3.19), we have

$$\deg(a_k)=\deg(a_{k-1})+\deg(\alpha)-\deg(\beta), \qquad k=r+1, ..., 2r$$

so that

$$\begin{aligned}\deg(a_k)&=\deg(a_r)+(k-r)(\deg(\alpha)-\deg(\beta))\\ &=r+(k-r)(\deg(\alpha)-\deg(\beta)), \qquad k=r+1, ..., 2r.\end{aligned} \tag{3.24}$$

In particular, we have for $k=2r$ in (3.24) $\deg(a_{2r})=r(\deg(\alpha)-s)$. Since $\deg(a_{2r})\geqslant\deg(\alpha)\geqslant 0$, $s\leqslant\deg(\alpha)<s+2$ so that $\deg(\alpha)$ is either s or $s+1$. If $\deg(\alpha)=s$, then $s=0$ and so $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS. If $\deg(\alpha)=s+1$, then we have by counting degrees on both sides of the equation (3.20)

$$\deg(h_k)=\deg(h_{k-1}),\qquad k=r+2, ..., 2r$$

so that

$$\deg(h_k)=r-1,\qquad k=r+1, ..., 2r. \tag{3.25}$$

If we set $g_k(x)=h_{k+2}(x)$, $k=r-1, ..., 2(r-1)$, then $\{g_k(x)\}_{r-1}^{2(r-1)}$ satisfy the condition (ii) in Theorem 3.1 with r replaced by $r-1$ and σ replaced by u_0 by (3.16) and (3.25). Hence, by Theorem 3.1 and induction hypothesis, $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS relative to u_0. ∎

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